Additional Material: Elastic Textures for Additive Fabrication

1 Derivation of Periodic Homogenization

As outlined in Section 5 of the paper, the homogenization process proceeds by considering the solution of (3), the displacement $\mathbf{u}^\epsilon$, as $\epsilon \to 0$. One approach in taking this limit is through two-scale asymptotic expansions [All02, All92].

Two-scale analysis. The two-scale method is based on the ansatz that for small ϵ, the family of solutions $\mathbf{u}^\epsilon$, parameterized by ϵ, can be written as

$$\mathbf{u}^\epsilon(\mathbf{x}) = \sum_{p=0}^{\infty} \epsilon^p \mathbf{u}_p \left(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}\right). \tag{A1}$$

Each function $\mathbf{u}_p(\mathbf{x}, \mathbf{y})$ separates its dependence on $\mathbf{x}$ (i.e. the smoothly varying, macroscopic part) from its dependence on $\mathbf{y} = \mathbf{x}/\epsilon$ (the microscopic fluctuations), and is constrained to be periodic in $\mathbf{y}$. Plugging the series into (3), collecting coefficients of ϵ^p terms, and identifying each coefficient of ϵ^p as an individual equation, yields a set of equations for $\mathbf{u}_p$

$$\epsilon^{-2}:\ -\nabla_{\mathbf{y}} \cdot [C(\mathbf{y}) : \varepsilon_{\mathbf{y}}(\mathbf{u}_0)] = \mathbf{0} \tag{A2}$$

$$\begin{aligned}\epsilon^{-1}:\ -\nabla_{\mathbf{y}} \cdot [C(\mathbf{y}) : (\varepsilon_{\mathbf{y}}(\mathbf{u}_1) + \varepsilon_{\mathbf{x}}(\mathbf{u}_0))] - \\ \nabla_{\mathbf{x}} \cdot [C(\mathbf{y}) : \varepsilon_{\mathbf{y}}(\mathbf{u}_0)] = \mathbf{0},\end{aligned} \tag{A3}$$

$$\begin{aligned}\epsilon^{0}:\ -\nabla_{\mathbf{y}} \cdot [C(\mathbf{y}) : (\varepsilon_{\mathbf{y}}(\mathbf{u}_2) + \varepsilon_{\mathbf{x}}(\mathbf{u}_1))] - \\ \nabla_{\mathbf{x}} \cdot [C(\mathbf{y}) : (\varepsilon_{\mathbf{y}}(\mathbf{u}_1) + \varepsilon_{\mathbf{x}}(\mathbf{u}_0))] = \mathbf{f},\end{aligned} \tag{A4}$$

where subscripts $\mathbf{x}$ and $\mathbf{y}$ signify partial differentiation with respect to the $\mathbf{x}$ and $\mathbf{y}$ parameters. Here we made repeated use of the chain rule, e.g. $\nabla \mathbf{u}_0 \left(\mathbf{x}, \frac{\mathbf{x}}{\epsilon}\right) = \nabla_{\mathbf{x}} \mathbf{u}_0 + \frac{1}{\epsilon} \nabla_{\mathbf{y}} \mathbf{u}_0$.

Equation (A2) is satisfied by a function independent of $\mathbf{y}$, $\mathbf{u}_0(\mathbf{x}, \mathbf{y}) \equiv \mathbf{u}(\mathbf{x})$. Using this, (A3) implies a linear relationship between $\varepsilon_{\mathbf{y}}(\mathbf{u}_1)$ and $\varepsilon_{\mathbf{x}}(\mathbf{u})$, which we can express with a rank 4 tensor F,

$$\varepsilon_{\mathbf{y}}(\mathbf{u}_1) = F : \varepsilon_{\mathbf{x}}(\mathbf{u}), \tag{A5}$$

mapping macroscopic strain to microscopic fluctuation strain.

Equation (A4) uniquely determines $\mathbf{u}_2$ from $\mathbf{u}$ and $\mathbf{u}_1$ up to a rigid offset if and only if a compatibility condition is met (the Fredholm alternative): the average of the left and right hand sides over the periodic cell Y must equal. Integrating both sides over Y, we see that the $\nabla_{\mathbf{y}}\cdot$ term disappears by the Divergence theorem and the periodicity of $C(\mathbf{y}) : (\varepsilon_{\mathbf{y}}(\mathbf{u}_2) + \varepsilon_{\mathbf{x}}(\mathbf{u}_1))$, leaving

$$-\nabla_{\mathbf{x}} \cdot \int_Y C(\mathbf{y}) : [\varepsilon_{\mathbf{y}}(\mathbf{u}_1) + \varepsilon_{\mathbf{x}}(\mathbf{u})] \, \mathrm{d}\mathbf{y} = |Y| \mathbf{f}.$$

Using (A5)

$$-\nabla_{\mathbf{x}} \cdot \left[\left(\frac{1}{|Y|} \int_Y C(\mathbf{y}) : F + C(\mathbf{y}) \, \mathrm{d}\mathbf{y} \right) : \varepsilon_{\mathbf{x}}(\mathbf{u}) \right] = \mathbf{f}, \tag{A6}$$

Comparing this with (4) from the paper implies

$$C^H = \frac{1}{|Y|} \int_Y C(\mathbf{y}) : F + C(\mathbf{y}) \, \mathrm{d}\mathbf{y} \tag{A7}$$

Local microscopic displacement. It remains to determine rank 4 tensor F appearing in C^H using (A3). Let e^{kl} denote the canonical basis for symmetric rank 2 tensors:

$$e^{kl} = \frac{1}{2}\left(\mathbf{e}_k \otimes \mathbf{e}_l + \mathbf{e}_l \otimes \mathbf{e}_k\right)$$

where $\mathbf{e}_k$ is the k^{th} canonical basis vector. Then the macroscopic strain can be decomposed as $\varepsilon_{\mathbf{x}}(\mathbf{u}) = [\varepsilon_{\mathbf{x}}(\mathbf{u})]_{kl} e^{kl}$. Notice if Y-periodic $\mathbf{w}^{kl}(\mathbf{y})$ satisfies:

$$-\nabla_{\mathbf{y}} \cdot \left(C(\mathbf{y}) : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right) = \mathbf{0}, \tag{A8}$$

then, by linearity, any $\mathbf{u}_1$ with $\varepsilon_{\mathbf{y}}(\mathbf{u}_1) = [\varepsilon_{\mathbf{x}}(\mathbf{u})]_{kl} \varepsilon_{\mathbf{y}}(\mathbf{w}^{kl})$ satisfies (A3):

$$-\nabla_{\mathbf{y}} \cdot \left(C(\mathbf{y}) : [\varepsilon_{\mathbf{y}}(\mathbf{u}_1) + \varepsilon_{\mathbf{x}}(\mathbf{u})]\right) = -[\varepsilon_{\mathbf{x}}(\mathbf{u})]_{kl} \left[\nabla_{\mathbf{y}} \cdot \left(C(\mathbf{y}) : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right)\right] = \mathbf{0}, \tag{A9}$$

implying

$$F_{pqkl} = [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl})]_{pq}. \tag{A10}$$

Plugging this into (A7), we get

$$C^H_{ijkl} = \frac{1}{|Y|} \int_Y C_{ijpq}(\mathbf{y}) [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl})]_{pq} + C_{ijkl}(\mathbf{y}) \, \mathrm{d}\mathbf{y}. \tag{A11}$$

Thus, once we know each "fluctuation displacement" $\mathbf{w}^{kl}$, we can compute the homogenized elasticity tensor with a simple integration over the base cell. We find these by solving the 6 cell problems

$$-\nabla_{\mathbf{y}} \cdot \left(C(\mathbf{y}) : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right) = \mathbf{0} \quad \text{in } Y, \tag{A12a}$$

$$\mathbf{w}^{kl}(\mathbf{y}) \; Y\text{-periodic}, \tag{A12b}$$

$$\int_\omega \mathbf{w}^{kl}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \mathbf{0}, \tag{A12c}$$

one for each canonical basis tensor e^{kl}. The last constraint is to fix the remaining translational degree of freedom; since we only care about strain $\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl})$, we can arbitrarily choose to enforce $\mathbf{0}$ average displacement over the microstructure geometry, ω.

In our single material setting,

$$C(\mathbf{y}) = \begin{cases} C^{\text{base}} & \text{if } \mathbf{y} \in \omega, \\ 0 & \text{otherwise} \end{cases}, \tag{A13}$$

meaning that the cell problem formulation above actually involves delta functions, and the displacements outside the microstructure can be arbitrary. However, we can avoid these problems by rephrasing the force balance as a PDE over the microstructure geometry only:

$$-\nabla_{\mathbf{y}} \cdot \left(C^{base} : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right) = \mathbf{0} \quad \text{in } \omega, \tag{A14a}$$

$$\hat{\mathbf{n}} \cdot \left(C^{base} : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right) = \mathbf{0} \quad \text{on } \partial\omega \setminus Y, \tag{A14b}$$

$$\mathbf{w}^{kl}(\mathbf{y}) \; Y\text{-periodic}, \tag{A14c}$$

$$\int_\omega \mathbf{w}^{kl}(\mathbf{y}) \, \mathrm{d}\mathbf{y} = \mathbf{0}. \tag{A14d}$$

Finally, we can rewrite the average stress form of the homogenized tensor, (A11), in an energy-like form that is more amenable to shape differentiation. First, note that (A11) can be rewritten as:

$$C^H_{ijkl} = \frac{1}{|Y|} \int_\omega e^{ij} : C : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}] \, \mathrm{d}\mathbf{y}. \tag{A15}$$

Notice that, for an arbitrary periodic function $\phi(\mathbf{y})$, integration by parts tells us:

$$\begin{aligned} \int_\omega \varepsilon_{\mathbf{y}}(\phi) : C : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}] \, \mathrm{d}\mathbf{y} = &- \int_\omega \phi \cdot \left[\nabla_{\mathbf{y}} \cdot \left(C : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right)\right] \, \mathrm{d}\mathbf{y} \\ &+ \int_{\partial\omega} \phi \cdot \left[\hat{\mathbf{n}} \cdot \left(C : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}]\right)\right] \, \mathrm{d}\mathbf{y} = 0. \end{aligned} \tag{A16}$$

The volume integral and the $\partial\omega \setminus \partial Y$ portion of the boundary integral vanished because $\mathbf{w}^{kl}$ solves the kl^{th} cell problem, and the $\partial\omega \cap \partial Y$ portion vanished due to periodicity. Taking $\phi = \mathbf{w}^{ij}$, this shows (A15) can be rewritten:

$$C^H_{ijkl} = \frac{1}{|Y|} \int_\omega [\varepsilon_{\mathbf{y}}(\mathbf{w}^{ij}) + e^{ij}] : C : [\varepsilon_{\mathbf{y}}(\mathbf{w}^{kl}) + e^{kl}] \,\mathrm{d}\mathbf{y}. \tag{A17}$$

2 Shape Derivative of C^H_{ijkl}

Due to its similarity to the self-adjoint compliance functional, (A17) has a surprisingly simple shape derivative.

Consider a perturbation of the shape's boundary, $\delta\omega$, caused by advecting the boundary with an infinitesimal velocity field $\mathbf{v}$. The resulting variation of C^H_{ijkl} for $ij \neq kl$ ($ij = kl$ gets an equivalent result by product rule):

$$\delta C^H_{ijkl} = \left\langle \frac{\partial C^H_{ijkl}}{\partial \omega}, \delta\omega \right\rangle + \left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{ij}}, \delta\mathbf{w}^{ij} \right\rangle + \left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{kl}}, \delta\mathbf{w}^{kl} \right\rangle.$$

Consider the linear functional $\left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{ij}}, \cdot \right\rangle$ on an arbitrary admissible perturbation of $\mathbf{w}^{ij}$ (periodic and with no rigid translation component), ϕ:

$$\left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{ij}}, \phi \right\rangle = \lim_{h\to 0} \frac{\mathrm{d}}{\mathrm{d}h} \frac{1}{|Y|} \int_\omega (e^{ij} + e(\mathbf{w}^{ij} + h\phi)) : C : (e^{kl} + e(\mathbf{w}^{kl})) \,\mathrm{d}\mathbf{y}$$

Differentiating under the integral and using the linearity of strain, this is:

$$\left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{ij}}, \phi \right\rangle = \frac{1}{|Y|} \int_\omega e(\phi) : C : (e^{kl} + e(\mathbf{w}^{kl})) \,\mathrm{d}\mathbf{y} = 0,$$

where we again used (A16). The same argument holds for $\left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{kl}}, \phi \right\rangle$, so we have

$$\left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{ij}}, \delta\mathbf{w}^{ij} \right\rangle = \left\langle \frac{\partial C^H_{ijkl}}{\partial \mathbf{w}^{kl}}, \delta\mathbf{w}^{kl} \right\rangle = 0$$

without solving an adjoint problem. Thus Reynold's transport theorem gives the full shape derivative:

$$\mathrm{d}C^H_{ijkl}[\mathbf{v}] := \delta C^H_{ijkl} = \left\langle \frac{\partial C^H_{ijkl}}{\partial \omega}, \delta\omega \right\rangle = \frac{1}{|Y|} \int_{\partial\omega} \left[(e^{ij} + e(\mathbf{w}^{ij})) : C : (e^{kl} + e(\mathbf{w}^{kl})) \right] \mathbf{v} \cdot \hat{\mathbf{n}} \,\mathrm{d}A(\mathbf{y}).$$

References

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[All02] Grégoire Allaire. *Shape Optimization by the Homogenization Method.* Number v. 146 in Applied Mathematical Sciences. Springer, 2002.